

EXTENDING OPEN FAMILIES IN NONMETRIC SPACES AND AN APPLICATION TO OVERLAYS

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Abstract. We show that a family \mathcal{U} of open subsets of a subset X of a hereditarily normal paracompact space P which is locally finite in P admits a locally finite open extension \mathcal{V} in P which is similar to \mathcal{U} . We apply this fact to prove an extension theorem for overlays over hereditarily paracompact spaces.

1. Introduction

Let P be a space and X be a subset of P . We are concerned with the problem of extending an open family \mathcal{U} in X to an open family \mathcal{V} in P so that whenever a finite subfamily of \mathcal{V} has a nonempty intersection then the corresponding finite subfamily of \mathcal{U} also has a nonempty intersection. If the last happens then we say that \mathcal{U} and \mathcal{V} are similar, see the beginning of Section 2 for more precise description. In case (P, d) is a metric space, Kuratowski's formula [6] $V_U = \{p \in P \mid d(p, U) < d(p, X \setminus U)\}$, $U \in \mathcal{U}$, yields a family $\mathcal{V} = \{V_U\}_{U \in \mathcal{U}}$ which is similar to \mathcal{U} . In this paper we consider the case of a hereditarily normal paracompact space P . Extending an old result due to Čech, we show that every open family \mathcal{U} of subsets of an arbitrary set $X \subset P$ which is locally finite in P admits a locally finite open extension \mathcal{V} which is similar to \mathcal{U} . (Previously Fox claimed a similar fact, however, his proof of [3, Lemma 5.5] contained a gap; see [5].)

Following an idea of Fox [3], we apply our result to prove an extension theorem for overlays. Let us recall that the concept of an overlay $e : \tilde{X} \rightarrow X$ is a shape-theory counterpart of a covering projection (introduced by Fox in [3]). Our extension theorem for overlays states that if X is an arbitrary subset of a hereditarily paracompact space P then, for an overlay $e : \tilde{X} \rightarrow X$, there exists an open neighborhood U of X in P and an overlay $\bar{e} : \tilde{U} \rightarrow U$ which extends e .

2. Extension of Covers

Let P be a space and X be a subset of P . Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ and $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ be families of subsets of X and P , respectively. The family \mathcal{V} is

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called an extension of the family \mathcal{U} if, for every $\alpha \in A$, we have that $V_\alpha \cap X = U_\alpha$. The families \mathcal{U} and \mathcal{V} are said to be similar provided, for every finite set of indices $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset A$, we have $U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_n} = \emptyset$ if and only if $V_{\alpha_1} \cap V_{\alpha_2} \cap \dots \cap V_{\alpha_n} = \emptyset$.

THEOREM 2.1. *Let P be a hereditarily normal paracompact space and X be a subset of P . Then, for every family $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of open subsets of X which is locally finite in P , there exists an open locally finite family $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ in P such that*

- (1) \mathcal{V} is an extension of \mathcal{U} ; and
- (2) \mathcal{U} and \mathcal{V} are similar.

The case when \mathcal{U} is finite is a result of Čech [1]; for completeness we enclose a proof of it (which slightly differs from the original one).

LEMMA 2.2. *Let X be a subset of a hereditarily normal space P . Every finite open family \mathcal{U} of subsets of X admits an open extension \mathcal{V} in P which is similar to \mathcal{U} .*

Proof. Denote by $r = r(\mathcal{U})$ the greatest cardinality of a subfamily $\mathcal{W} \subset \mathcal{U}$ with a nonempty intersection. We will use induction on r .

Suppose $r(\mathcal{U}) = 1$ for some cover \mathcal{U} ; this means that \mathcal{U} is pairwise disjoint. One can easily see that it is enough to consider the case when \mathcal{U} has two elements. However, it is well-known that two disjoint open subsets of a subset X of a hereditarily normal space P can be extended to disjoint open subsets of P (see [2, Thm. 2.1.7]). This settles the case of $r = 1$.

Suppose that the assertion of our lemma holds for all families \mathcal{U} with $1 \leq r(\mathcal{U}) < k$. Let $\mathcal{U} = \{U_0, U_2, \dots, U_{n-1}\}$ be a family of open subsets of X with $r(\mathcal{U}) = k$. If $n = k$ then $U_0 \cap U_1 \cap \dots \cap U_{n-1} \neq \emptyset$. In this case an arbitrary open extension \mathcal{V} of \mathcal{U} works. So, we can assume that $n > k$. Let $[n]^k$ denote the family of all subsets of the set $n = \{0, \dots, n-1\}$ of the cardinality k . Consider the family \mathcal{W} consisting of all sets of the form $W_I = \bigcap \{U_i \mid i \in I\}$ where $I \in [n]^k$. For each such W_I and index $i \notin I$ we have that $W_I \cap U_i = \emptyset$. For each $I \in [n]^k$ and $i \in n \setminus I$ we choose open sets $W(I, i)$ and $U(I, i)$ in P that extend W_I and U_i , respectively, and such that $W(I, i) \cap U(I, i) = \emptyset$. We let $W_I^* = \bigcap \{W(I, i) \mid i \in n \setminus I\}$ and $U_i^* = \bigcap \{U(I, i) \mid I \in [n]^k, i \notin I\}$. We can additionally assume that $W_I^* \subset U_i^*$ provided $i \in I$. Let $W_0 = \bigcup \mathcal{W}$. Define $X' = X \setminus W_0 \subset P \setminus W_0 = P'$. Write $\mathcal{U}' = \{X' \cap U_0, X' \cap U_1, \dots, X' \cap U_{n-1}\}$ and notice that \mathcal{U}' is an open family in X' with $r(\mathcal{U}') < k$. By the inductive assumption, there exists an open family \mathcal{W}' in P' which extends \mathcal{U}' and is similar to \mathcal{U}' . We can additionally assume that each element $W'_i \in \mathcal{W}'$ extending $X' \cap U_i$ is contained in U_i^* . Finally, for every $U_i \in \mathcal{U}$, define $V_i = W'_i \cup \bigcup \{W_I^* \mid I \in [n]^k \text{ and } i \in I\}$. The family $\{V_0, V_1, \dots, V_{n-1}\}$ is as required. \square

Proof of Theorem 2.1. Since \mathcal{U} is locally finite in P , there exists an open cover \mathcal{W} of P such that each element $W \in \mathcal{W}$ intersects only finitely many of elements \mathcal{U} . It follows that $\text{cl}(W)$, the closure of W in P has the same property. By the

paracompactness of P we can additionally assume that \mathcal{W} is locally finite (in P). Clearly, the family $\{\text{cl}(W) \mid W \in \mathcal{W}\}$ is also locally finite.

Fix $W \in \mathcal{W}$. Consider all U_α whose intersection with $\text{cl}(W)$ is nonempty. There are only finitely many of such U_α , say $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$. Apply Lemma 2.2 to the space P , the subset $X \cap \text{cl}(W)$, and the family $\mathcal{U}_W = \{U_{\alpha_1} \cap \text{cl}(W), U_{\alpha_2} \cap \text{cl}(W), \dots, U_{\alpha_n} \cap \text{cl}(W)\}$ to find a family $\mathcal{O}_W = \{O_{\alpha_1}, O_{\alpha_2}, \dots, O_{\alpha_n}\}$ of open subsets in P such that, for each α_i ($i = 1, 2, \dots, n$), $O_{\alpha_i} \cap \text{cl}(W) = U_{\alpha_i} \cap \text{cl}(W)$, and the family \mathcal{O}_W is similar to \mathcal{U}_W . For each element $U = U_{\alpha_i}$, let U_W denote the corresponding O_{α_i} . We let $U_W = \emptyset$ if $U_\alpha \cap \text{cl}(W) = \emptyset$.

Now, fix $U = U_\alpha$. Look at the family $\{F(U, W) \mid W \in \mathcal{W}\}$, where $F(U, W) = \text{cl}(W) \setminus U_W$. Since $\{F(U, W) \mid W \in \mathcal{W}\}$ is inscribed in locally finite family $\{\text{cl}(W) \mid W \in \mathcal{W}\}$, it is locally finite as well. Consequently, the set $F_U = \bigcup \{F(U, W) \mid W \in \mathcal{W}\}$ is a closed set in P . We set V_α to be $P \setminus F_U$. We claim that the family $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ is as required.

It follows from the definition that each V_α is an open subset of P . We will show that if a finite subfamily $\{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ of \mathcal{V} has a nonempty intersection then the corresponding subfamily $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ of \mathcal{U} has a nonempty intersection. Fix $x \in P$. Let $x \in V_{\alpha_1} \cap V_{\alpha_2} \cap \dots \cap V_{\alpha_n}$ (the intersection may reduce to a single set). Select $W \in \mathcal{W}$ such that $x \in W$. Consequently, we have $x \in V_{\alpha_1} \cap V_{\alpha_2} \cap \dots \cap V_{\alpha_n} \cap W$. We now conclude that, for each $U = U_{\alpha_i}$ ($i = 1, 2, \dots, n$), $x \notin F(U, W)$. It follows that $x \in U_W$ for each such U . Hence, we have that $x \in \bigcap \{U_W \mid U = U_{\alpha_i}\} \neq \emptyset$. Since the last family is similar to the family $\{U_{\alpha_1} \cap \text{cl}(W), U_{\alpha_2} \cap \text{cl}(W), \dots, U_{\alpha_n} \cap \text{cl}(W)\}$, we conclude that $U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_n} \cap \text{cl}(W) \neq \emptyset$. In particular, we have that $U_{\alpha_1} \cap U_{\alpha_2} \cap \dots \cap U_{\alpha_n} \neq \emptyset$. In a similar way we obtain that, for every $\alpha \in A$, $V_\alpha \cap X \subseteq U_\alpha$. On the other hand, for every $U = U_\alpha$, V_α contains U because every $F(U, W)$ is disjoint from U . This shows that, for every $\alpha \in A$, $V_\alpha \cap X = U_\alpha$.

Moreover, the family \mathcal{V} is locally finite in P . It is clear that each $W \in \mathcal{W}$ intersects only finitely many of V_α (those for which U_α intersects $\text{cl}(W)$). \square

COROLLARY 2.3. *Let X be a closed subset of a hereditarily normal paracompact space P . Then, every open locally finite family \mathcal{U} in X admits an open locally finite extension \mathcal{V} in P so that \mathcal{V} is similar to \mathcal{U} .*

Proof. Notice that \mathcal{U} is locally finite in P , and apply Theorem 1. \square

COROLLARY 2.4. *Let X be a subset of a hereditarily paracompact space P . Then, every open locally finite family $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ in X admits an open extension \mathcal{V} in P so that \mathcal{V} is locally finite in $\bigcup \mathcal{V}$ and is similar to \mathcal{U} .*

Proof. There exists an open neighborhood V_0 of X in P so that the family \mathcal{U} is locally finite in V_0 . Now, we can apply Theorem 1 to X regarded as a subset of a hereditarily normal paracompact space V_0 . \square

3. Extending overlays

Let us recall that a covering projection $e : \tilde{X} \rightarrow X$ is a map such that every point $x \in X$ has an open neighborhood U such that $e^{-1}(U)$ is a disjoint union of open

subsets \tilde{U}^α and, for each α , $e|_{\tilde{U}^\alpha}$ is a homeomorphism of \tilde{U}^α onto U . We do not assume here that X is locally connected. The following modification of this notion is due to Fox [3].

Definition. Let \tilde{X} and X be two Hausdorff spaces, $\mathcal{M} = \{M_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of X , and $e: \tilde{X} \rightarrow X$ be a map. A collection $\tilde{\mathcal{M}} = \{\tilde{M}_\lambda^\alpha\}_{\lambda \in \Lambda}^{\alpha \in A_\lambda}$ of subsets of \tilde{X} is said to **lie evenly** over the collection \mathcal{M} if

- (1) $e^{-1}(M_\lambda) = \bigcup_{\alpha \in A_\lambda} \tilde{M}_\lambda^\alpha$ for each index $\lambda \in \Lambda$;
- (2) for each $\alpha \in A_\lambda$, the set \tilde{M}_λ^α is open in $e^{-1}(M_\lambda)$;
- (3) for each $\alpha \in A_\lambda$, the set \tilde{M}_λ^α is mapped by $e|_{\tilde{M}_\lambda^\alpha} = e|_{\tilde{M}_\lambda^\alpha}$ homeomorphically onto M_λ ; and
- (4) if $M_\lambda \cap M_{\lambda'} \neq \emptyset$ then, for each $\alpha \in A_\lambda$, the set \tilde{M}_λ^α meets exactly one of the sets $\tilde{M}_{\lambda'}^\beta$, $\beta \in A_{\lambda'}$, (in particular, we have that $\tilde{M}_\lambda^\alpha \cap \tilde{M}_{\lambda'}^\beta = \emptyset$ whenever $\alpha \neq \beta$).

The map $e: \tilde{X} \rightarrow X$ will be called an **overlay** if \tilde{X} has an open cover $\tilde{\mathcal{M}}$ that lies evenly over some open cover \mathcal{M} of X .

Notice that from the fact that $\tilde{\mathcal{M}}$ is an open cover and condition (3) it follows that e is continuous.

Let η_λ denote the cardinality of the family $\{\tilde{M}_\lambda^\alpha \mid \alpha \in A_\lambda\}$. Thus A_λ can be identified with the set of ordinals $\eta_\lambda = \{\alpha \mid 0 \leq \alpha < \eta_\lambda\}$. In general η_λ need not be the same as $\eta_{\lambda'}$, but of course they must be the same whenever M_λ and $M_{\lambda'}$ intersect. In such a case $\eta_\lambda = \eta_{\lambda'}$ and a permutation $\omega_{\lambda\lambda'}$ of η_λ is determined as follows:

$$\omega_{\lambda\lambda'}(\alpha) = \beta,$$

where $\tilde{M}_\lambda^\alpha \cap \tilde{M}_{\lambda'}^\beta \neq \emptyset$. Note that $\omega_{\lambda\lambda}$ is the identity permutation. Moreover, we have that $\omega_{\lambda'\lambda} = \omega_{\lambda\lambda'}^{-1}$, and $\omega_{\lambda\lambda'}\omega_{\lambda'\lambda''} = \omega_{\lambda\lambda''}$ whenever $M_\lambda \cap M_{\lambda'} \cap M_{\lambda''} \neq \emptyset$.

The following fact presumably belongs to mathematical folklore; we present a proof of it because we could not find one in the literature.

LEMMA 3.1. *Let $e: \tilde{X} \rightarrow X$ be a covering projection. If X is a hereditarily paracompact space, then \tilde{X} is a hereditary paracompact space.*

Proof. Let \tilde{U} be an open subspace of \tilde{X} . We must show that \tilde{U} is paracompact. Since $e: \tilde{X} \rightarrow X$ is a covering projection the set $e(\tilde{U}) = U$ is an open subset of X , and there is an open cover $\{M_\lambda\}$ of U so that $e^{-1}(M_\lambda)$ is the disjoint union of an open family $\{\tilde{M}_\lambda^\alpha\}$ of \tilde{X} and $e|_{\tilde{M}_\lambda^\alpha}: \tilde{M}_\lambda^\alpha \rightarrow M_\lambda$ is a homeomorphism. Since X is a hereditarily paracompact space we can find a closed locally finite cover \mathcal{V} of U inscribed in the cover $\{M_\lambda\}$. For a given $V \in \mathcal{V}$, being a disjoint union of paracompact spaces, the space $U_V = e^{-1}(V) \cap \tilde{U}$ is paracompact. Now, since \tilde{U} is covered by $\{U_V \mid V \in \mathcal{V}\}$, a locally finite closed family of paracompact spaces, it is paracompact (see [2, Thm. 5.1.34]). \square

If $\bar{e}: \bar{U} \rightarrow U$ is an overlay and $X \subset U$, then letting $\bar{X} = \bar{e}^{-1}(X)$, the map $e = \bar{e}|_{\bar{X}}: \bar{X} \rightarrow X$ is also an overlay. We will say that e is a restriction of \bar{e} and that \bar{e} is an extension of e .

THEOREM 3.2. *If X is a subset of a hereditarily paracompact space P and $e: \bar{X} \rightarrow X$ is an overlay, then for some open neighborhood U of X in P there exists an overlay $\bar{e}: \bar{U} \rightarrow U$ that is an extension of e . Moreover, \bar{U} is hereditarily paracompact.*

Proof. We will follow the lines of the proof of [3, Theorem 5.2].

Since $e: \bar{X} \rightarrow X$ is an overlay there is an open covering $\{M_\lambda\}$ of X over which lies evenly an open covering $\{\bar{M}_\lambda^\alpha\}$ of \bar{X} . Since X is a paracompact space it may be assumed that $\{M_\lambda\}$ is locally finite in X . By Corollary 2.4, there exists an open family $\{V_\lambda\}$ in P which is similar to the cover $\{M_\lambda\}$ and such that $\{V_\lambda\}$ is locally finite in the union of $\{V_\lambda\}$.

Let $U = \bigcup_\lambda V_\lambda$; this is an open neighborhood of X in P . For each $\lambda \in \Lambda$ and $\alpha \in \eta_\lambda$, let V_λ^α be a topological space homeomorphic to V_λ , and let r_λ^α be a homeomorphism of V_λ^α onto V_λ . In the discrete union V of all the spaces V_λ^α , $\alpha \in \eta_\lambda$, $\lambda \in \Lambda$, let us introduce the following equivalence relation:

- (1) if $p \in V_\lambda^\alpha$ and $p' \in V_{\lambda'}^\beta$, then $p \approx p'$ if and only if $r_\lambda^\alpha(p) = r_{\lambda'}^\beta(p')$ and $\omega_{\lambda\lambda'}(\alpha) = \beta$.

The relation \approx is symmetric and reflexive. The fact that, for any triple $\lambda, \lambda', \lambda''$, if $M_\lambda \cap M_{\lambda'} \cap M_{\lambda''} = \emptyset$ then $V_\lambda \cap V_{\lambda'} \cap V_{\lambda''} = \emptyset$ yields the transitivity of \approx . Hence the quotient space $\bar{U} = V/\approx$ is well-defined. Let us denote by q the quotient map of V onto \bar{U} and by q_λ^α the restriction $q|_{V_\lambda^\alpha}$ of q to V_λ^α . Clearly, q_λ^α maps V_λ^α homeomorphically onto a subset \bar{V}_λ^α of \bar{U} . Since $q^{-1}(\bar{V}_\lambda^\alpha) = \bigcup_{\lambda, \beta} (r_\lambda^\beta)^{-1}(V_\lambda)$, the set

\bar{V}_λ^α is open in \bar{U} . We infer that $\bar{U} = \bigcup_{\lambda, \alpha} \bar{V}_\lambda^\alpha$.

For any point \bar{p} of \bar{U} there is at least one point p of V for which $q(p) = \bar{p}$, and if p and p' are two such points then $p \approx p'$. Hence, if we define $\bar{e}(\bar{p})$ to be the image under an appropriate r_λ^α of a point of $q^{-1}(\bar{p})$ the definition is unambiguous. Obviously $\bar{e}q_\lambda^\alpha = r_\lambda^\alpha$ for every λ, α . Since, by the definition, $\bar{e}: \bar{U} \rightarrow U$ maps each \bar{V}_λ^α topologically onto V_λ , it is easily seen that \bar{e} is an overlay. We will show that \bar{e} is an extension of e up to some homeomorphism of \bar{X} .

Consider a point \bar{x} of \bar{X} . It belongs to at least one of the set \bar{M}_λ^α . Therefore the point $x = e(\bar{x})$ must belong to M_λ and hence to V_λ . Let p_λ^α denote the point of V_λ^α that is mapped by r_λ^α onto x , and let $\bar{p} = q(p_\lambda^\alpha)$. Let us define $f(\bar{x})$ to be the point \bar{p} . This is unambiguous definition for if $\bar{x} \in \bar{M}_\lambda^\alpha \cap \bar{M}_{\lambda'}^\beta$, then $x \in M_\lambda \cap M_{\lambda'}$ and $p_\lambda^\alpha \approx p_{\lambda'}^\beta$. Since $f|_{\bar{M}_\lambda^\alpha}$ is obviously a homeomorphism of \bar{M}_λ^α onto a subset of \bar{V}_λ^α , and $\{M_\lambda\}$ is locally finite, it is easy to show that f is a homeomorphism onto $f(\bar{X})$. Then, finally we note that

$$\bar{e}(f(\bar{x})) = \bar{e}(\bar{p}) = \bar{e}(q_\lambda^\alpha(p_\lambda^\alpha)) = r_\lambda^\alpha(p_\lambda^\alpha) = x = e(\bar{x}).$$

It can be easily observed that $f(\bar{X}) = \bar{e}^{-1}(X)$.

Since P is hereditarily paracompact space, the space U is also hereditarily paracompact. By Lemma 3.1, the space \tilde{U} is hereditarily paracompact. \square

Remark. Let X be a closed subset of a hereditarily normal paracompact space P . Then, every overlay $e : \tilde{X} \rightarrow X$ admits an extension to an overlay $\bar{e} : \tilde{U} \rightarrow U$, where U is some open neighborhood of X in P . (To obtain this, replace Corollary 2.4 by Corollary 2.3 in the proof of Theorem 3.2.) \square

Our original project was to use Theorem 3.2 for an extension of the classical lifting property of covering projections to overlays $e : \tilde{X} \rightarrow X$. This was accomplished by Fox [3,4] for the case of metrizable X . Actually we are able to carry out Fox's program for some hereditarily paracompact spaces X . However, after having learned that Mroziński [7] (using a different approach) has recently obtained the lifting property for overlays over an arbitrary space X , we decided not to include our partial results on that subject herein.

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